Kolmogorov Scaling in Truncated 3-D Euler Flows

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ABSTRACT: Kolmogorov-like turbulence is obtained during an intermediate regime of the spontaneous relaxation of (time-reversible) spectrally-truncated Euler equations towards absolute equilibrium. Dissipative effects are estimated near the equilibrium using Monte-Carlo methods and Fluctuation Dissipation relations. Scaling laws are derived for the wavenumbers at which dissipation becomes relevant. Possible experimental investigations of this new regime of turbulence, using visco-elastic materials, are suggested.

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The dynamics of spectrally truncated time reversible nonlinear equations has already been studied in the particular cases of 1-D Burgers-Hopf models [1] and 2-D quasi-geostrophic flows [2]. A central point in these studies was the nature of the statistical equilibrium that is achieved at large times [3] . Several equilibria are *a priori* possible because both (truncated) 1-D Burgers-Hopf and 2-D quasi-geostrophic flow models admit, besides the energy, a number of additional conserved quantities.

The purpose of the present Letter is to study the dynamics of spectrally truncated 3-D incompressible Euler flows. This problem is of a different nature because (except for helicity that identically vanishes for the flows considered here) there is no known additional conserved quantity [4] and the equilibrium is thus unique. The central problem in truncated 3-D Eulerian dynamics is that of the mechanism of relaxation toward that equilibrium.

The main result of this Letter is that large-scale Eulerian dynamics together with small-scale statistical equilibration seem to be enough to generate Kolmogorov scaling at intermediate scales. Note that the short-time (i.e. spectrally converged) truncated Eulerian dynamics has been extensively studied [5] in order to obtain numerical evidence for or against blowup [6] of the original (untruncated) Euler equation.

The three-dimensional incompressible Euler equations for a fluid of unit density,

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p \, , \tag{1}$$

$$\nabla\cdot\mathbf{v} = 0 \, , \tag{2}$$

are solved numerically using standard [7] periodic pseudo-spectral methods with resolution N. The solutions are dealiased by spectrally truncating the modes for which at least one wave-vector component exceeds $N/3$ (thus a 1024^3 run is truncated at $k_{\max} = 341$). This method amounts to a Galerkin truncation of the original equations and exactly conserves the energy.

Time marching is done with a second-order leapfrog finite-difference scheme, even and odd time-steps are periodically re-coupled using fourth-order Runge-Kutta.

The energy spectrum is defined by averaging $\hat{\mathbf{v}}(\mathbf{k}', t)$ (the spatial Fourier transform of the solution to Eq. (1)) on spherical shells of width $\Delta k = 1$,

$$E(k,t) = \frac{1}{2} \sum_{k-\Delta k/2<|\mathbf{k}'|<k+\Delta k/2} |\hat{\mathbf{v}}(\mathbf{k}',t)|^2 \, . \tag{3}$$

It is well known [3, 8] that the truncated equations (1), (2) admit statistically stationary exact solutions, the so-called absolute equilibria, with Gaussian distribution f^* and energy spectra $E(k) = \text{cte} \times k^2$. Furthermore, the temporal fluctuations around the equilibria are related to the equilibrium correlation functions by a Fluctuation Dissipation Theorem (FDT). Indeed, let $S(t)$ denote the equilibrium response-functions, defined by

$$S^{ij}_{(\mathbf{k},\mathbf{k}')}(t) = \int f^*(\hat{v}_0) \frac{\partial \hat{v}^i(t,\mathbf{k})}{\partial \hat{v}^j(0,\mathbf{k})} D\hat{v}_0. \tag{4}$$

In this equation, $\hat{\mathbf{v}}(0,\mathbf{k})$ represents an initial velocity field, $\hat{\mathbf{v}}(t,\mathbf{k})$ represents the velocity field at time t obtained from $\hat{\mathbf{v}}(0,\mathbf{k})$ by the equations of motion and $\hat{v}_0$ stands for the set of all $\hat{v}^i(0,\mathbf{k})$ components; $D\hat{v}_0$ is the usual Lebesgue measure in $\hat{v}_0$-space. Let also $\Delta(t)$ denote the equilibrium correlations, defined by:

$$\Delta^{ij}_{(\mathbf{k},\mathbf{k}')}(t) = \int f^*(\hat{v}_0) \hat{v}^i(t,\mathbf{k}) \hat{v}^j(0,\mathbf{k}) D\hat{v}_0. \tag{5}$$

The FDT [9, 10] states that $S(t)$ is proportional, with constant coefficients, to the time-derivative of Δ. Consequently, the time-evolutions of S and Δ are characterized by the same characteristic time, say τ_C.

To study relaxation toward absolute equilibrium, we use the so-called Taylor-Green [11] single–mode initial data $u^{TG} = \sin(x)\cos(y)\cos(z)$, $v^{TG} = u^{TG}(y,-x,z)$,

$w^{TG} = 0$. Symmetries are employed in a standard way [12] to reduce memory storage and speed up computations. Runs were made with $N = 256$, 512 and 1024.

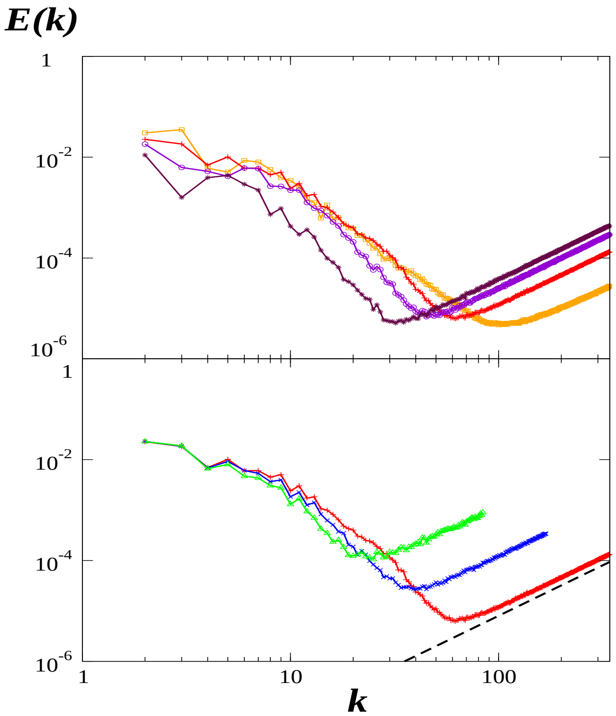

FIG. 1: Energy spectra, top: resolution 1024^3 at $t = (6.5, 8, 10, 14)$ ($\diamond$,+,$\circ$,$*$); bottom: resolutions 256^3 (triangle $\triangle$), 512^3 (cross $\times$) and 1024^3 (cross +) at $t = 8$, the dashed line indicates k^2 scaling.

Figure 1 displays the time evolution (top) and resolution dependence (bottom) of the energy spectra. It is apparent that a wavevector $k_{\rm min}$ (such that $E(k) \geq E(k_{\rm min})$) spontaneously appears in the flow. The modes with $k > k_{\rm min}$ appear to be in absolute equilibrium (see the dashed line at the bottom of the figure). Defining the thermalized (or dissipated) energy $E_{\rm th}$ by

$$E_{\rm th}(t) = \sum_{k_{min}<k} E(k,t) \ , \tag{6}$$

the time evolutions of $k_{\rm min}$ and $E_{\rm th}$ are presented on figure 2. It is apparent on the figure that, for all resolutions, $k_{\rm min}$ decreases and $E_{\rm th}$ increases with time and that, for all times, $k_{\rm min}$ increases and $E_{\rm th}$ decreases with the resolution.

A first hint for Kolmogorov behavior is given by the energy dissipation rate

$$\varepsilon(t) = \frac{{\rm d}E_{\rm th}(t)}{{\rm d}t} \ . \tag{7}$$

Indeed, perhaps one of the main quantitative results of this paper is the excellent agreement of the energy dissipation rate shown on figure 3 (top) with the corresponding data in the viscous TG flow (see reference [12], figure 7 and reference [4], figure 5.12). Both the time for maximum energy dissipation $t_{max} \simeq 8$ and the value of the dissipation rate at that time $\epsilon(t_{max}) \simeq 1.5\,10^{-2}$ are in

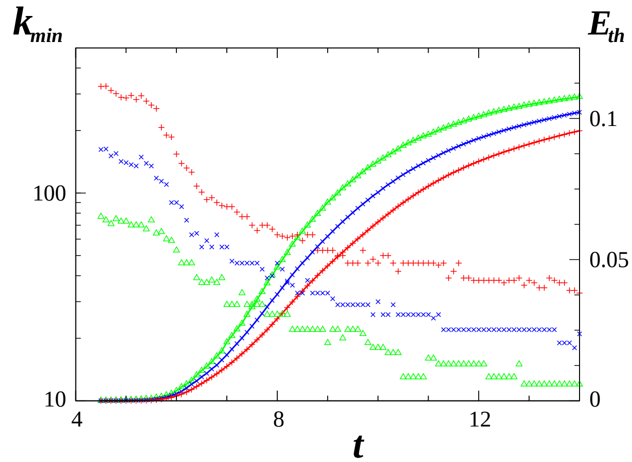

FIG. 2: Time evolution of $k_{\rm min}$ (left vertical axis) and $E_{\rm th}$ (right vertical axis) at resolutions 256^3 (triangle $\triangle$), 512^3 (cross $\times$) and 1024^3 (cross +).

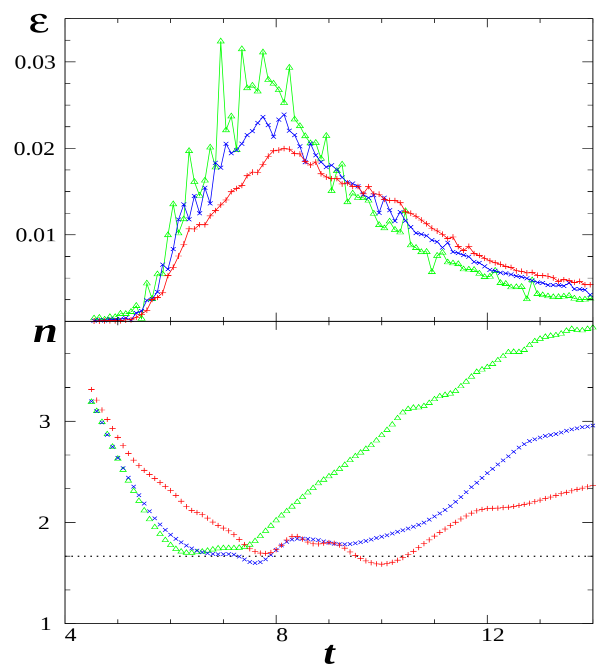

FIG. 3: Temporal evolution of, top: energy dissipation ε; bottom: k^{-n} inertial range prefactor n at resolutions 256^3 (triangle $\triangle$), 512^3 (cross $\times$) and 1024^3 (cross +).

quantitative agreement.

A confirmation for Kolmogorov behavior around t_{max} is displayed on figure 3 (bottom). The value of the inertial-range prefactor n, obtained by a low-k least square fit of the log of the energy spectrum with the function $\mathrm{cte} - n\log(k)$, is close to $5/3$ (horizontal dashed line) when $t \simeq t_{max}$.

Assuming Kolmogorov scaling $E(k) \sim \varepsilon^{2/3}k^{-5/3}$ in the $k < k_{min}$ range and absolute equilibrium $E(k) \sim 3k^2 E_{\rm th}/k_{\rm max}^3$ in the $k > k_{min}$ range, one obtains [10] a first estimation k_m for the observed wavenumber k_{min}.

$$k_{\rm m} \sim \left(\frac{\varepsilon}{E_{\rm th}^{3/2}}\right)^{2/11} k_{\rm max}^{9/11} \, . \tag{8}$$

The ratio k_{min}/k_m is displayed on figure 4. It is seen to be reasonably constant on the figure.

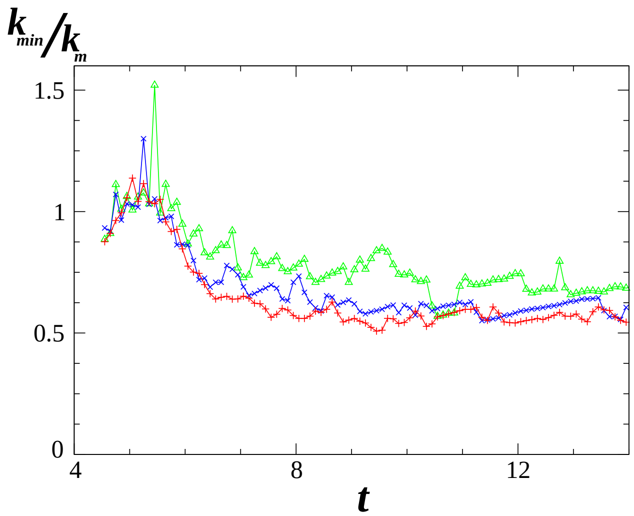

FIG. 4: Time evolution of the ratio k_{min}/k_m at resolutions 256^3 (triangle Δ), 512^3 (cross $\times$) and 1024^3 (cross $+$).

However a detailed inspection of figure 1 strongly suggests that the inertial-range spectra are not pure power laws and that some kind of dissipative effects are present. One may therefore try to relate the observed value of k_{min} to the relaxation time τ_R, defined as the characteristic time-scale of the response function (4) around the high-k absolute equilibrium. The FDT (see text below Eq. (5)) states that $\tau_R = \tau_C$, where τ_C is the equilibrium correlation time. We have therefore estimated the values of τ_C by performing (general periodic) Monte-Carlo computations of the (shell averaged) correlation function (5). The correlation time τ_C associated to wavenumber k is found [10] to obey the simple scaling law

$$\tau_C = \frac{C}{k\sqrt{E_{\rm th}}} \, . \tag{9}$$

The Monte-Carlo values for the constant C are displayed on figure 5, where it is apparent that $C \sim 2.5$, provided that scale separation ($k \ll k_{\rm max}$) holds. This amounts to replacing the standard Navier-Stokes relation $\varepsilon(k,t) = \nu k^2 E(k,t)$ by $\varepsilon(k,t) = \bar{\nu}|k|E(k,t)$, where $\bar{\nu} = \sqrt{E_{\rm th}}/C$ and $\varepsilon(k,t) = -\partial E(k,t)/\partial t$ is the dissipation spectral density.

Assuming that this dissipation takes place in a range of width αk_d around k_d , we estimate the total dissipation $\varepsilon \sim \bar{\nu} k_d E(k_d)\alpha k_d$. This, together with $E(k_d) \sim k_d^2 E_{\rm th}/k_{\rm max}^3$ yields the dissipative estimate for $k_{\rm min}$

$$k_d \sim \left(\frac{\varepsilon}{E_{\rm th}^{3/2}}\right)^{1/4} k_{\rm max}^{3/4} \, . \tag{10}$$

Note that (8) supposes a pure power law throughout the inertial range while (10) is a dissipative estimate. They can be consistent with one another only if $k_m > k_d$. One finds

$$k_m/k_d \sim \left(\frac{\varepsilon}{E_{\rm th}^{3/2}}\right)^{-3/44} k_{\rm max}^{3/44} \, . \tag{11}$$

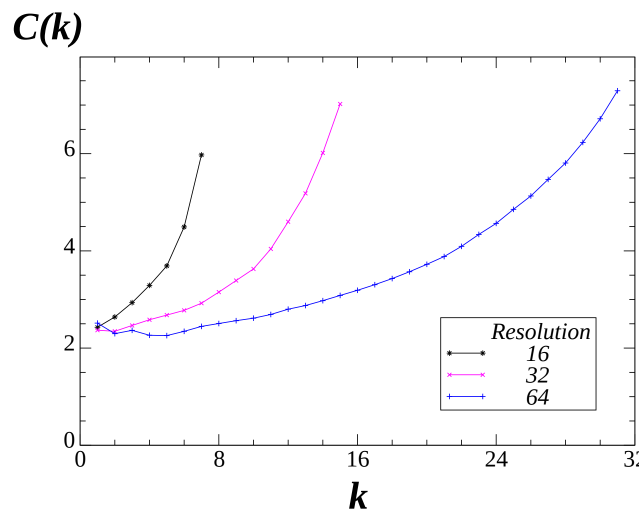

FIG. 5: Monte-Carlo estimation of relaxation time around the absolute equilibrium (see Eq. (9))

The assumptions thus appear to be consistent in the limit $k_{\rm max} \to \infty$, provided that $\varepsilon/{E_{\rm th}}^{3/2}$ does not depend drastically on $k_{\rm max}$. However, because of the limited range of variation of ε, $E_{\rm th}$, and $k_{\rm max}$ it was not possible to check numerically the exponents of (11).

Besides its dissipative effects, the spectral truncation mainly acts as a barrier that blocks the flow of energy to the small scales. Note that it might be possible, using visco-elastic materials such as wormlike micelles [13], to experimentally generate such a blocking [14].

In this context, the truncated Euler equations appears as a minimal model of turbulence. Let us mention that Kolmogorov scaling has already been observed in non-viscous systems, in the context of (compressible) low-temperature superfluid turbulence [15, 16, 17]. This behavior has also been reproduced using simple Biot-Savart vortex methods[18].

Note that spontaneous small-scale equilibration happening in isolated systems, such as the one studied in the present Letter, should not be confused with equilibration resulting from contact with an external thermostat. Indeed the reversible dynamics of the isolated system generates spontaneously a (time and initial conditions dependent) temperature.

In summary, we have observed Kolmogorov-like turbulence in truncated Eulerian dynamics. Scaling laws have been obtained for the dissipative effects that spontaneously appear in this time-reversible system. However, it remains an open problem whether experiments using visco-elastic materials and/or higher resolution runs will confirm these laws.

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